

# **A note on Morita modules and quotient rings**

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## A note on Morita modules and quotient rings

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In this note, every ring has an identity 1 and every module over a ring is unital. A ring extension  $A/B$  means  $B$  is a subring of  $A$  containing  $1_A$  (the identity of  $A$ ) and a ring homomorphism means such one that the image of 1 is 1.

A homomorphism will be usually written at the opposite side of the scalar.

An  $A'-A$ -bimodule  ${}_A U_A$  is said to be a Morita module if  $U_A$  is a progenerator and  $A' = \text{End}(U_A)$  ([7] p.98).

We consider ring extensions  $A/B$  and  $A'/B'$  which have Morita modules  ${}_A U_A$  and  ${}_{B'} V_B$ .

**Lemma 1.** ([7], p.111). Under the above situations, the following statements are equivalent.

- (1) There exists a  $B'-A$ -isomorphism  $\phi: {}_{B'} V \otimes_B A_A \rightarrow {}_{B'} U_A$ .
- (2) There exists an  $A'-B$ -isomorphism  $\phi': {}_{A'} A' \otimes_{B'} V_B \rightarrow {}_{A'} U_B$ .

In this case, the equation  $\phi(v \otimes 1) = \phi'(1 \otimes v)$  holds for any  $v \in V$ .

Proof. (1) $\Rightarrow$ (2): Since  ${}_{B'} V$  is finitely generated and projective, we have

$$\begin{aligned} {}_{A'} A' \otimes_{B'} V_B &\cong {}_{A'} \text{Hom}(U_A, U_A) \otimes_{B'} V_B \cong {}_{A'} \text{Hom}(\text{Hom}({}_{B'} V, {}_{B'} U_A), U_A)_B \\ &\cong {}_{A'} \text{Hom}(\text{Hom}({}_{B'} V, {}_{B'} V \otimes_B A_A), U_A)_B \cong {}_{A'} \text{Hom}(\text{Hom}({}_{B'} V, {}_{B'} V) \otimes_B A_A, U_A)_B \\ &\cong {}_{A'} \text{Hom}(A_A, U_A)_B \cong {}_{A'} U_B. \end{aligned}$$

Hence an isomorphism  $\phi'$  exists and the above correspondences are given by

$$\begin{aligned} \sum a'_j \otimes v_j &\rightarrow \sum (x \rightarrow a'_j x) \otimes v_j \rightarrow (g \rightarrow \sum a'_j \cdot (v_j)g) \\ &\rightarrow (h \rightarrow \sum a'_j \cdot \phi((v_j)h)) = ((v \rightarrow v \otimes a) \rightarrow ua) \leftarrow (1 \otimes a \rightarrow ua) \\ &\leftarrow (a \rightarrow ua) \leftarrow u = \phi'(\sum a'_j \otimes v_j). \end{aligned}$$

So, for any  $a \in A$ ,  $\sum a'_j \cdot \phi(v_j \otimes 1) \cdot a = \sum a'_j \cdot \phi(v_j \otimes a) = ua = \phi'(\sum a'_j \otimes v_j) \cdot a = \sum a'_j \cdot \phi'(1 \otimes v_j) \cdot a$ .

Especially,  $\phi(v \otimes 1) = \phi'(1 \otimes v)$  for any  $v \in V$ .

(2) $\Rightarrow$ (1): This is similarly proved.

If the conditions of Lemma 1 are satisfied, then ring extensions  $A/B$  and  $A'/B'$  are said to be Morita equivalent ([7] p.111, [5] p.74).

**Proposition 2.** If Lemma 1 is satisfied, then the following statements are equivalent.

- (1) The map  ${}_A A \otimes_B A_A \rightarrow {}_A A_A$  ( $x \otimes y \rightarrow xy$ ) is an isomorphism.
- (2) The map  ${}_{A'} A' \otimes_{B'} V \otimes_B A_A \rightarrow {}_{A'} U_A$  ( $a' \otimes v \otimes a \rightarrow a' \phi(v \otimes a) = \phi'(a' \otimes v)a$ ) is an isomorphism.
- (3) The map  ${}_{A'} A' \otimes_{B'} A'_{A'} \rightarrow {}_{A'} A'_{A'}$  ( $x' \otimes y' \rightarrow x'y'$ ) is an isomorphism.

Proof. (1) $\Rightarrow$ (2): By Lemma 1, we have

$$\begin{aligned} {}_{A'} A' \otimes_{B'} V \otimes_B A_A &\cong {}_{A'} U \otimes_B A_A \cong {}_{A'} U \otimes_A A \otimes_B A_A \cong {}_{A'} U \otimes_A A_A \cong {}_{A'} U_A. \\ a' \otimes v \otimes a &\rightarrow \phi'(a' \otimes v) \otimes a \rightarrow \phi'(a' \otimes v) \otimes 1 \otimes a \rightarrow \phi'(a' \otimes v) \otimes a \rightarrow \phi'(a' \otimes v)a \\ \phi'^{-1}(u) \otimes 1 &\leftarrow u \otimes 1 \leftarrow u \otimes 1 \otimes 1 \leftarrow u \otimes 1 \leftarrow u. \end{aligned}$$

(2) $\Rightarrow$ (3): Since  ${}_{A'} U_A \cong {}_{A'} A' \otimes_{B'} V \otimes_B A_A \cong {}_{A'} A' \otimes_{B'} U_A \cong {}_{A'} A' \otimes_{B'} A'_{A'} \otimes_{A'} U_A$

$$u \rightarrow \phi'^{-1}(u) \otimes 1 \rightarrow \sum a'_j \otimes \phi(v_j \otimes 1) \rightarrow \sum a'_j \otimes 1 \otimes \phi(v_j \otimes 1) \text{ where } \phi'^{-1}(u) = \sum a'_j \otimes v_j$$

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$\sum \phi'(x' \otimes w_k) s_k \leftarrow x' \otimes \phi^{-1}(y't) \leftarrow x' \otimes y't \leftarrow x' \otimes y' \otimes t$  where  $\phi^{-1}(y't) = \sum w_k \otimes s_k$   
and  $U_A$  is finitely generated and projective, we have

$$\begin{aligned} {}_A A' \otimes_{B'} A'_{A'} &\cong {}_A A' \otimes_{B'} A' \otimes_A \text{Hom}(U_A, U_A)_{A'} \cong {}_A \text{Hom}(U_A, A' \otimes_{B'} A' \otimes_A U_A)_{A'} \\ &\cong {}_A \text{Hom}(U_A, U_A)_{A'} \cong {}_A A'_{A'}. \\ x' \otimes y' &\rightarrow x' \otimes 1 \otimes (u \rightarrow y'u) \rightarrow (t \rightarrow x' \otimes 1 \otimes y't = x' \otimes y' \otimes t) \\ &\rightarrow (t \rightarrow \sum \phi'(x' \otimes w_k) s_k = \sum x' \phi(w_k \otimes s_k) = x'y't) \rightarrow x'y'. \end{aligned}$$

(3) $\Rightarrow$ (2): By Lemma 1, we have  ${}_A A' \otimes_{B'} V \otimes_B A_A \cong {}_A A' \otimes_{B'} U_A \cong {}_A A' \otimes_{B'} A' \otimes_A U_A$   
 $\cong {}_A U_A$   $a' \otimes v \otimes a \rightarrow a' \otimes \phi(v \otimes a) \rightarrow a' \otimes 1 \otimes \phi(v \otimes a) \rightarrow a' \otimes \phi(v \otimes a)$ .

(2) $\Rightarrow$ (1): Since  ${}_A U_A \cong {}_A A' \otimes_{B'} V \otimes_B A_A \cong {}_A U \otimes_B A_A \cong {}_A U \otimes_A A \otimes_B A_A$   
 $u \rightarrow \phi^{-1}(u) \otimes 1 \rightarrow u \otimes 1 \rightarrow u \otimes 1 \otimes 1$

and  ${}_A U$  is finitely generated and projective, we have

$$\begin{aligned} {}_A A \otimes_B A_A &\cong {}_A \text{Hom}({}_A U, {}_A U) \otimes_A A \otimes_B A_A \cong {}_A \text{Hom}({}_A U, {}_A U \otimes_A A \otimes_B A_A) \\ &\cong {}_A \text{Hom}({}_A U, {}_A U) \cong {}_A A_A \\ x \otimes y &\rightarrow (u \rightarrow ux) \otimes 1 \otimes y \rightarrow (u \rightarrow ux \otimes 1 \otimes y) \rightarrow (u \rightarrow uxy) \rightarrow xy. \end{aligned}$$

**Lemma 3.** If Lemma 1 is satisfied, then

- (1)  ${}_A \text{Hom}(U_A, A_A)_{A'} \cong {}_A \text{Hom}({}_A U, {}_A A')_{A'}$  ( $f \rightarrow f'$ )
- (2)  ${}_B \text{Hom}(V_B, B_B)_{B'} \cong {}_B \text{Hom}({}_B V, {}_B B')_{B'}$  ( $g \rightarrow g'$ )

For the above correspondences, the equations  $y \cdot f(x) = (y)f' \cdot x$  and  $z \cdot g(w) = (z)g' \cdot w$  hold for any  $x, y \in U$  and  $w, z \in V$ .

Proof. (1) We have  ${}_A \text{Hom}(U_A, A_A)_{A'} \cong {}_A \text{Hom}(U_A, \text{Hom}({}_A U, {}_A U)_{A'})_{A'}$   
 $\cong {}_A \text{Hom}({}_A U, {}_A \text{Hom}(U_A, U_A)_{A'})_{A'} \cong {}_A \text{Hom}({}_A U, {}_A A')_{A'}$

$$\begin{aligned} f &\rightarrow h_1: (x_1 \rightarrow (y_1 \rightarrow y_1 \cdot f(x_1))) \\ &\rightarrow h_2: (y_2 \rightarrow (x_2 \rightarrow (y_2)[h_1(x_2)] = y_2 \cdot f(x_2)) \rightarrow f' \end{aligned}$$

and  $(y_2)f' \cdot x_2 = [(y_2)f'](x_2) = y_2 \cdot f(x_2)$ .

(2) is similarly proved.

The modules which were defined in (1) and (2) of Lemma 3 will be written as  $U^*$  and  $V^*$  respectively.

**Lemma 4.** If Lemma 1 is satisfied, then

- (1)  ${}_A U^*_{B'} \cong {}_A \text{Hom}(V_B, A_B)_{B'}$
- (2)  ${}_B U^*_{A'} \cong {}_B \text{Hom}({}_B V, {}_B A')_{A'}$
- (3)  ${}_A U \otimes_B V^*_{B'} \cong {}_A A'_{B'}$
- (4)  ${}_B V \otimes_B U^*_{A'} \cong {}_B A'_{A'}$
- (5)  ${}_B V^* \otimes_{B'} U_A \cong {}_B A_A$
- (6)  ${}_A U^* \otimes_{B'} V_B \cong {}_A A_B$

Proof. (1) By Lemma 1, we have  ${}_A U^*_{B'} \cong {}_A \text{Hom}(U_A, A_A)_{B'} \cong {}_A \text{Hom}(V \otimes_B A_A, A_A)_{B'}$   
 $\cong {}_A \text{Hom}(V_B, \text{Hom}(A_A, A_A)_{B'})_{B'} \cong {}_A \text{Hom}(V_B, A_B)_{B'}$ .

(2) By Lemma 1, we have  ${}_B U^*_{A'} \cong {}_B \text{Hom}({}_A U, {}_A A')_{A'} \cong {}_B \text{Hom}({}_A A' \otimes_{B'} V, {}_A A')_{A'}$   
 $\cong {}_B \text{Hom}({}_B V, {}_B \text{Hom}({}_A A', {}_A A'))_{A'} \cong {}_B \text{Hom}({}_B V, {}_B A')_{A'}$ .

(3) Since  $V_B$  is finitely generated and projective, by Lemma 1, we have

$$\begin{aligned} {}_A U \otimes_B V^*_{B'} &\cong {}_A U \otimes_B \text{Hom}(V_B, B_B)_{B'} \cong {}_A \text{Hom}(V_B, U \otimes_B B_B)_{B'} \cong {}_A \text{Hom}(V_B, U_B)_{B'} \\ &\cong {}_A \text{Hom}(V_B, A' \otimes_{B'} V_B)_{B'} \cong {}_A A' \otimes_{B'} \text{Hom}(V_B, V_B)_{B'} \cong {}_A A'_{B'}. \end{aligned}$$

(4) Since  $U_A$  is finitely generated and projective, by Lemma 1, we have

$${}_B V \otimes_B U^*_{A'} \cong {}_B V \otimes_B \text{Hom}(U_A, A_A)_{A'} \cong {}_B \text{Hom}(U_A, V \otimes_B A_A)_{A'} \cong {}_B \text{Hom}(U_A, U_A)_{A'} \cong {}_B A'_{A'}.$$

(5) By Lemma 1, we have  ${}_B V^* \otimes_{B'} U_A \cong {}_B V^* \otimes_{B'} V \otimes_B A_A \cong {}_B A_A$ .

(6) Since  ${}_A U$  is finitely generated and projective, by Lemma 1, we have

$${}_A U^* \otimes_{B'} V_B \cong {}_A \text{Hom}({}_A U, {}_A A') \otimes_{B'} V_B \cong {}_A \text{Hom}({}_A U, {}_A A' \otimes_{B'} V) \cong {}_A \text{Hom}({}_A U, {}_A U)_{B'} \cong {}_A A_B.$$

**Proposition 5.** If Lemma 1 is satisfied, then we have

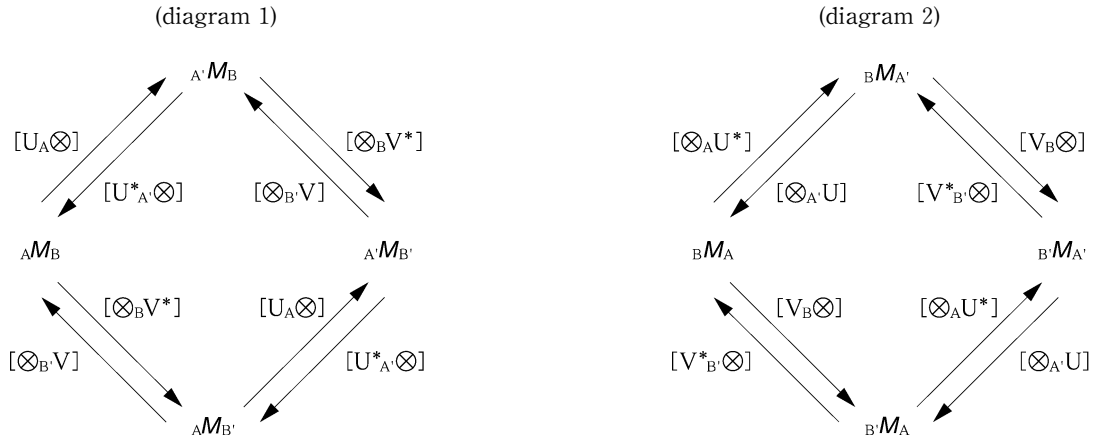
(1)  ${}_B A$  is flat if and only if  ${}_{B'} A'$  is flat.

(2)  $A_B$  is flat if and only if  $A'_{B'}$  is flat.

Proof. (1) If  ${}_B A$  is flat, then  ${}_{B'} V \otimes_B A \cong {}_{B'} U$  is flat and hence  ${}_{B'} U \otimes_A U^* \cong {}_{B'} A'$  is flat. If  ${}_{B'} A'$  is flat, then  ${}_{B'} U \otimes_A U^*$  is flat and by Lemma 4 (5),  ${}_B V^* \otimes {}_{B'} U \otimes_A U^* \cong {}_B A \otimes_A U^* \cong {}_B U^*$  is flat. Hence  ${}_B U^* \otimes_{A'} U \cong {}_B A$  is flat.

(2) is similarly proved by Lemma 4 (6).

Let  ${}_A \mathbf{M}_B$  be the isomorphism classes of  $A$ - $B$ -bimodules. We consider the following two diagrams.



where, for example, the map  $[U_A \otimes]$  is defined by  $[U_A \otimes][{}_A X_B] = [{}_A U \otimes {}_A X_B]$ .

If the conditions of Lemma 1 and Proposition 2 are satisfied, in each isomorphism classes of diagram 1 and diagram 2, a binary multiplication are defined with a left identity and a right identity respectively.

For example, the multiplication in  ${}_A \mathbf{M}_B$  is defined by  $[{}_A X_B][{}_A Y_B] = [{}_A X \otimes_B Y_B]$ , and since  ${}_A A \otimes_B Y_B \cong {}_A A \otimes_B A \otimes_A Y_B \cong {}_A A \otimes_A Y_B \cong {}_A Y_B$ ,  $[{}_A A_B]$  is a left identity. The multiplication in  ${}_A \mathbf{M}_{B'}$  is defined by  $[{}_A X_{B'}][{}_A Y_{B'}] = [{}_A U \otimes_A U^* \otimes_{A'} X \otimes_B U^* \otimes_{A'} Y_{B'}] = [{}_A X \otimes_B U^* \otimes_{A'} Y_{B'}]$  and  $[{}_A U_B]$  is a left identity. By Lemma 4 (4), the multiplication in  ${}_A \mathbf{M}_{B'}$  is defined by  $[{}_A X_{B'}][{}_A Y_{B'}] = [{}_A U \otimes_A U^* \otimes_{A'} X \otimes_{B'} V \otimes_B U^* \otimes_{A'} Y_{B'} \otimes_{B'} V \otimes_B V^*_{B'}] = [{}_A X \otimes_{B'} A' \otimes_{A'} Y_{B'}] = [{}_A X \otimes_{B'} Y_{B'}]$  and  $[{}_A A'_{B'}]$  is a left identity. The multiplication in  ${}_A \mathbf{M}_{B'}$  is defined by  $[{}_A X_{B'}][{}_A Y_{B'}] = [{}_A X \otimes_{B'} V \otimes_B Y_{B'}]$  and by Lemma 4 (1),  $[{}_A U^*_{B'}] = [{}_A A \otimes_B V^*_{B'}]$  is a left identity.

We can consider similarly in the diagram 2, and  $[{}_B A_A]$  is a right identity of  ${}_B \mathbf{M}_A$ .

**Proposition 6.** Maps of diagram 1 (resp. diagram 2) are bijective and preserve the multiplications and left (resp. right) identities.

Let  ${}_{B'} V_B$  be a Morita module. It is well known that there exists a one-to-one correspondence between the set of (two sided) ideals  $\{J'\}$  of  $B'$  and the set of (two sided) ideals  $\{J\}$  of  $B$  under the correspondence  $J' \rightarrow \{b \in B \mid Vb \subseteq J'V\}$  and  $\{b' \in B' \mid b'V \subseteq VJ\} \leftarrow J$  ([2], p.6).

In this note, we will always assume that  $B$  is a right Noetherian and right hereditary ring. That is,  $B$  is right Noetherian and every right ideal of  $B$  is projective. Then  $B'$  is also a right Noetherian and right hereditary ring ([3], p.378).

An ideal  $J$  of  $B$  is called dense as a right ideal if  $bJ = 0$  then  $b = 0$  for any  $b \in B$  ([6], p.96).

Let  $\mathbf{D} = \{J \mid J \text{ is an ideal of } B \text{ and dense as a right ideal of } B\}$ . Then,

- (1) for ideals  $J$  and  $K$  of  $B$ , if  $J \in \mathbf{D}$  and  $J \subseteq K$ , then  $K \in \mathbf{D}$ ,
- (2) for ideals  $J$  and  $K$  of  $B$ , if  $J \in \mathbf{D}$  and  $K \in \mathbf{D}$ , then  $JK \in \mathbf{D}$ .

That is,  $\mathbf{D}$  is a filter.

Let  $A = \varinjlim_{\mathbf{D}} \text{Hom}(J_B, B_B)$  (a ring of right quotients of  $B$ ) ([1]).  $B$  is canonically a subring of  $A$ .

**Lemma 7.** For any  $J, K \in \mathbf{D}$ , we have

$${}_B \text{Hom}(J_B, B_B) \otimes_B \text{Hom}(K_B, B_B)_B \cong {}_B \text{Hom}(KJ_B, B_B)_B.$$

Proof. Since  $K_B$  is finitely generated and projective, we have

$$\begin{aligned} {}_B \text{Hom}(J_B, B_B) \otimes_B \text{Hom}(K_B, B_B)_B &\cong {}_B \text{Hom}(K_B, \text{Hom}(J_B, B_B) \otimes_B B_B)_B \\ &\cong {}_B \text{Hom}(K_B, \text{Hom}(J_B, B_B)_B)_B \cong {}_B \text{Hom}(K \otimes_B J_B, B_B)_B \\ &\cong {}_B \text{Hom}(KJ_B, B_B)_B \\ &\quad h_1 \otimes h_2 \rightarrow (x_1 \rightarrow h_1 \otimes h_2(x_1)) \\ &\quad \rightarrow (x_1 \rightarrow h_1 \cdot h_2(x_1)) \rightarrow (x_1 \otimes x_2 \rightarrow h_1[h_2(x_1) \cdot x_2]) \\ &\quad \rightarrow (x_1 \cdot x_2 \rightarrow [h_1 \cdot h_2](x_1 \cdot x_2)) = h_1 \cdot h_2. \end{aligned}$$

**Proposition 8.** Under the above situations, we have the map  $A \otimes_B A \rightarrow A$  ( $a_1 \otimes a_2 \rightarrow a_1 a_2$ ) is an isomorphism and  ${}_B A$  is flat.

Proof. Since any element of  $\mathbf{D}$  is finitely generated and projective, by Lemma 7,

$$\begin{aligned} A \otimes_B A &= \varinjlim_{\mathbf{D}} \text{Hom}(J_B, B_B) \otimes_B \varinjlim_{\mathbf{D}} \text{Hom}(K_B, B_B) \cong \varinjlim_{\mathbf{D} \times \mathbf{D}} \text{Hom}(K \otimes_B J, B_B) \\ &\cong \varinjlim_{\mathbf{D} \times \mathbf{D}} \text{Hom}(KJ_B, B_B) \cong \varinjlim_{\mathbf{D}} \text{Hom}(I_B, B_B) = A. \end{aligned}$$

Let  $0 \rightarrow X_B \rightarrow Y_B$  is exact. Since  $0 \rightarrow \text{Hom}(J_B, X_B) \rightarrow \text{Hom}(J_B, Y_B)$  is exact and  $J_B$  is finitely generated and projective,  $0 \rightarrow X \otimes_B \text{Hom}(J_B, B_B) \rightarrow Y \otimes_B \text{Hom}(J_B, B_B)$  is exact. Hence,  $0 \rightarrow X \otimes_B \varinjlim_{\mathbf{D}} \text{Hom}(J_B, B_B) \rightarrow Y \otimes_B \varinjlim_{\mathbf{D}} \text{Hom}(J_B, B_B)$  is exact.

**Lemma 9.** Let  $J \in \mathbf{D}$  and  $J' = \{b' \in B' \mid b'V \subseteq VJ\}$  be the ideal of  $B'$  which corresponds to  $J$ . Then,  $J'$  is dense as a right ideal of  $B'$  and

$$\begin{aligned} {}_{B'} V \otimes_B \text{Hom}(J_B, B_B) \otimes_B V^*_{B'} &\cong {}_{B'} \text{Hom}(J'_B, B'_B)_{B'} \\ \tau \otimes \delta \otimes \xi &\rightarrow (y \rightarrow \sum [\tau \cdot \{\delta \cdot \xi(v_j)\}(x_j)] \psi'_j) \end{aligned}$$

where  $J' \cong {}_{B'} V \otimes_B J \otimes_B V^*_{B'} = {}_{B'} V \otimes_B J \otimes_B \text{Hom}(B'_B, V_B)_{B'}$  ( $y \rightarrow \sum v_j \otimes x_j \otimes \psi'_j$ ) ( $y \in J'$ ,  $v_j \in V$ ,  $x_j \in J$ ,  $\psi'_j \in V^*$ ). In this case,  $y = \sum (v_j x_j) \psi'_j$ .

Proof. Since  $V_B$  is finitely generated and projective,

$$\begin{aligned} J' &\cong {}_{B'} \text{Hom}(V_B, VJ_B)_{B'} \cong {}_{B'} VJ \otimes_B \text{Hom}(V_B, B_B)_{B'} \\ &\cong {}_{B'} V \otimes_B J \otimes_B \text{Hom}(V_B, B_B)_{B'} \cong {}_{B'} V \otimes_B J \otimes_B \text{Hom}(B'_B, V_B)_{B'}, \\ &\cong {}_{B'} \text{Hom}(V^*_{B'}, {}_B JV^*)_{B'} \\ y &\rightarrow (v \rightarrow yv) \rightarrow \sum v_j x_j \otimes \psi_j \\ &\rightarrow \sum v_j \otimes x_j \otimes \psi_j \rightarrow \sum v_j \otimes x_j \otimes \psi'_j \\ &\rightarrow (g \rightarrow \sum g(v_j) x_j \psi_j). \end{aligned}$$

In this case, by Lemma 3, for any  $v \in V$ ,  $yv = \sum v_j \cdot x_j \cdot \psi_j(v) = \sum (v_j \cdot x_j) \psi'_j \cdot v$ .

Hence  $y = \sum (v_j x_j) \psi'_j$ . Moreover, since  $[\sum g(v_j) x_j \psi_j](v) = \sum g(v_j) x_j \psi_j(v) = g(\sum v_j x_j \psi_j(v)) = g(\sum (v_j x_j) \psi'_j v) = g(yv) = (gy)(v)$ , we have  $\sum g(v_j) x_j \psi_j = gy$ .

Let  $b'J' = 0$ . For any  $v \in V$  and  $w \in JV^*$ , we can define the map  $\eta_{v,w}: {}_B V^* \rightarrow {}_B JV^*$  ( $g \rightarrow g(v)w$ ). Then, for any  $g \in V^*$ , we have  $0 = (g)[b' \eta_{v,w}] = (gb') \eta_{v,w} = (gb')(v)w = g(b'v)w$ . Since  ${}_B JV^*$  is faithful,  $g(b'v) = 0$ . Hence  $b'v = 0$  and  $b' = 0$ . Further,

$$\begin{aligned}
& {}_{B'}V \otimes_B \text{Hom}(J_B, B_B) \otimes_B V^*_{B'} = {}_{B'}V \otimes_B \text{Hom}(J_B, B_B) \otimes_B \text{Hom}(V_B, B_B)_{B'} \\
& \cong {}_{B'}V \otimes_B \text{Hom}(V_B, \text{Hom}(J_B, B_B)_{B'}) \cong {}_{B'}V \otimes_B \text{Hom}(V \otimes_B J_B, B_B)_{B'} \\
& \cong {}_{B'}\text{Hom}(V \otimes_B J_B, V_B)_{B'} \cong {}_{B'}\text{Hom}(V \otimes_B J_B, \text{Hom}(V^*_{B'}, B'_{B'})_{B'}) \\
& \cong {}_{B'}\text{Hom}(V \otimes_B J \otimes_B V^*_{B'}, B'_{B'})_{B'} \cong {}_{B'}\text{Hom}(J'_{B'}, B'_{B'})_{B'}.
\end{aligned}$$

**Theorem 10.** Let  $A = \varinjlim_{\mathbf{D}} \text{Hom}(J_B, B_B)$  (a ring of right quotients of  $B$ ),  $U = V \otimes_B A$ , and  $A' = \text{End}(U_A)$ .

And let  $\mathbf{D}' = \{J' \subseteq B' \mid J' \text{ is an ideal of } B' \text{ and dense as a right ideal}\}$ . Then

$A' \cong \varinjlim_{\mathbf{D}'} \text{Hom}(J'_{B'}, B'_{B'})$  (a ring of right quotient of  $B'$ ) and  ${}_{B'}A'$  is flat.

Proof. By Lemma 9, we have

$$\begin{aligned}
& {}_{B'}\varinjlim_{\mathbf{D}'} \text{Hom}(J'_{B'}, B'_{B'})_{B'} \cong {}_{B'}V \otimes_B \varinjlim_{\mathbf{D}} \text{Hom}(J_B, B_B) \otimes_B V^*_{B'} \\
& \cong {}_{B'}V \otimes_B A \otimes_B V^*_{B'} = {}_{B'}U \otimes_B \text{Hom}(V_B, B_B)_{B'} \\
& \cong {}_{B'}\text{Hom}(V_B, U_B)_{B'} \\
& \cong {}_{B'}\text{Hom}(V_B, A' \otimes_{B'} V_B)_{B'} \cong {}_{B'}A' \otimes_{B'} \text{Hom}(V_B, V_B)_{B'} \cong {}_{B'}A'_{B'}.
\end{aligned}$$

Then, in the above isomorphisms, we will show that 1 of  $\varinjlim_{\mathbf{D}'} \text{Hom}(J'_{B'}, B'_{B'})$  corresponds to 1 of  $A'$ . Since  $V_B$  is finitely generated and projective, there exist

$$\begin{aligned}
& \tau_k \in V \text{ and } \xi_k \in \text{Hom}(V_B, B_B) \text{ such that for any } v \in V, v = \sum \tau_k \cdot \xi_k(v). \text{ Then,} \\
& [1] = [(y \rightarrow \sum_{j,k} [\tau_k \cdot \xi_k(v_j) \cdot x_j] \psi'_j = y)] \leftarrow \sum \tau_k \otimes [1] \otimes \xi_k \\
& \rightarrow \sum \tau_k \otimes 1 \otimes \xi_k = \sum (\tau_k \otimes 1) \otimes \xi_k \\
& \rightarrow (v \rightarrow \sum (\tau_k \otimes 1) \cdot \xi_k(v) = \sum \tau_k \cdot \xi_k(v) \otimes 1 = v \otimes 1_A = \phi'(a' \otimes v)) \\
& \leftarrow (v \rightarrow a' \otimes v) \leftarrow a' \otimes 1 \leftarrow a'.
\end{aligned}$$

In this case, by Lemma 1, for any  $a \in A$ ,  $v \otimes a = (v \otimes 1_A)a = \phi'(a' \otimes v)a = a' \phi'(1_A \otimes v)a = a'(v \otimes 1_A)a = a'(v \otimes a)$ . Hence  $a' = 1$ . That is, the isomorphism  $\varinjlim_{\mathbf{D}'} \text{Hom}(J'_{B'}, B'_{B'})_{B'} \rightarrow A'$  is an isomorphism as rings.

The fact that  ${}_{B'}A'$  is flat follows from Proposition 8 and Proposition 5.

**Proposition 11.** In Theorem 10, we have

$$\{M'_{B'} \mid M'_{B'} \cong M \otimes_B V^*_{B'} \text{ for some } M_B \text{ such that } M \otimes_B A = 0\} = \{M'_{B'} \mid M' \otimes_{B'} A' = 0\}.$$

Proof. Let  $M \otimes_B A = 0$ . Since  ${}_{B'}V$  and  $U_A$  are finitely generated and projective, by Lemma 4 (2),  $M \otimes_B V^*_{B'} \otimes_{B'} A' \cong M \otimes_B \text{Hom}({}_{B'}V, {}_{B'}B') \otimes_{B'} A' \cong M \otimes_B \text{Hom}({}_{B'}V, {}_{B'}A') \cong M \otimes_B U^* \cong M \otimes_B \text{Hom}(U_A, A_A) \cong \text{Hom}(U_A, M \otimes_B A_A) = 0$ .

Convesely, let  $M' \otimes_{B'} A' = 0$  and put  $M_B = M' \otimes_{B'} V_B$ . Then, we have  $M \otimes_B A = M' \otimes_{B'} V \otimes_B A \cong M' \otimes_{B'} U \cong M' \otimes_{B'} A' \otimes_{A'} U = 0$  and  $M' \otimes_{B'} V \otimes_B V^*_{B'} \cong M'_{B'}$ .

(\*) In this case,  $\mathbf{T} = \{M_B \mid M \otimes_B A = 0\}$  is a hereditary torsin class,

$$\mathbf{F} = \{\mathcal{O}_B \mid \mathcal{O}_B \text{ is a right ideal of } B \text{ such that } \mathcal{O}_B A = A\}$$

$$= \{\mathcal{O}_B \mid \mathcal{O}_B \text{ is a right ideal of } B \text{ such that } J \subseteq \mathcal{O}_B \text{ for some } J \in \mathbf{D}\}$$

is a topology and  $A \cong \varinjlim_{\mathbf{F}} \text{Hom}(\mathcal{O}_B, B_B)$  ([8], p.78).

$$\begin{aligned}
\text{Moreover, } \mathbf{T}' &= \{M'_{B'} \mid M'_{B'} \cong M \otimes_B V^*_{B'} \text{ for some } M_B \in \mathbf{T}\} \\
&= \{M'_{B'} \mid M' \otimes_{B'} A' = 0\}
\end{aligned}$$

is a hereditary torsion class,

$$\mathbf{F}' = \{\mathcal{O}'_{B'} \mid \mathcal{O}'_{B'} \text{ is a right ideal of } B' \text{ such that } \mathcal{O}'_{B'} A' = A'\}$$

$$= \{\mathcal{O}'_{B'} \mid \mathcal{O}'_{B'} \text{ is a right ideal of } B' \text{ such that } J' \subseteq \mathcal{O}'_{B'} \text{ for some } J' \in \mathbf{D}'\}$$

and  $A' \cong \varinjlim_{\vec{F}} \text{Hom}(\mathbb{Q}'_{B'}, B'_{B'})$  ([4], p.663).

## 要 約

森田加群に関連して、ある種の商環の森田同値性について調べた。

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